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NEW STABILITY ESTIMATES FOR THE INVERSE MEDIUM PROBLEM WITH INTERNAL DATA

MOURAD CHOULLI† AND FAOUZI TRIKI‡

ABSTRACT. A major problem in solving multi-waves inverse problems is the presence of critical points where the collected data completely vanishes. The set of these critical points depend on the choice of the boundary conditions, and can be directly determined from the data itself. To our knowledge, in the most existing stability results, the boundary conditions are assumed to be close to a set of CGO solutions where the critical points can be avoided. We establish in the present work new weighted stability estimates for an electro-acoustic inverse problem without assumptions on the presence of critical points. These results show that the Lipschitz stability far from the critical points deteriorates near these points to a logarithmic stability.

Mathematics subject classification : 35R30.

Key words : Multi-Wave Imaging, Critical points, Helmholtz equation, Hybrid Inverse Problems, Electro-Acoustic, Internal Data.

1. INTRODUCTION

Recently, number of works [ABGNS, ABGS, AGNS, ACGRT, BS, BU, SW] have developped a mathematical framework for new biomedical imaging modalities based on multi-wave probe of the medium. The objective is to stabilize and improve the resolution of imaging of biological tissues.

Different kinds of waves propagate in biological tissues and carry on informations on its properties. Each one of them is more sensitive to a specific physical parameter and can be used to provide an accurate image of it. For example, ultrasonic waves has influence on the density, electric waves are sensitive to the conductivity and optical waves impact the optical absorption. Imaging modalities based on a single wave are known to be ill-posed and suffer from low resolution [BT, SU]. Furthermore, the stability estimates related to these modalities are mostly logarithmic, i.e. an infinitesimal noise in the measured data may be exponentially amplified and may give rise to a large error in the computed solution [SU, BT]. One promising way to overcome the intrinsic limitation of single wave imaging and provide a stable and accurate reconstruction of physical parameters of a biological tissue is to combine different wave-imaging modalities [BBMT, ACGRT, Ka, Ku].

A variety of multi-wave imaging approaches are being introduced and studied over the last decade. The term multi-wave refers to the fact that two types of physical waves are used to probe the medium under study. Usually, the first wave is sensitive to the contrast of the desired parameter, the other types can carry the information revealed by the first type of waves to the boundary of the medium where measurements can be taken. There are different types of waves interaction that can be used to produce a single imaging system with best contrast and resolution properties of the two waves [FT]: the interaction of the first wave with the tissue can generate a second kind of wave ; a first wave that carries information about the contrast of the desired parameter can be locally modulated by a second wave that has better spatial resolution ; a fast propagating of the first wave can be used to acquire a spatio-temporal sequence of the propagation of a slower transient wave. Typically, the inversion procedure proceeds in two steps. Informations are retrieved from the waves of the second type by measurement on the boundary of the medium. The first inversion usually takes the form of a well-posed linear inverse source problem and provide internal data for the waves of the first type that are sensitive to the contrast of the desired physical parameter of the medium. The second step consists in recovering the values of this parameter from the acquired internal data.

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Here, we consider the peculiar case of electric measurements under elastic perturbations, where the medium is probed with acoustic waves while making electric boundary measurements [ABCTF, ACGRT, BS, BoT]. The associated internal data consists in the pointwise value of the electric energy density. In this paper, like in [Tr, BK], we focus on the second inversion, that is, to reconstruct the desired physical parameter of the medium from internal data. We refer the readers to the papers [ABCTF, ACGRT, BM] for the modelization of the coupling between the acoustic and electric waves and the way to get such internal data.

We introduce the mathematical framework of our inverse problem. On a bounded domain Ω of \mathbb{R}^n , $n = 2, 3$, with boundary Γ , we consider the second order differential operator

$$L_q = \partial_i(a^{ij}\partial_j \cdot) + q.$$

Following is a list of assumptions that may be used to derive the main results in the next section.

a1. The matrix $(a^{ij}(x))$ is symmetric for any $x \in \Omega$.

a2. (Uniform ellipticity condition) There is a constant $\lambda > 0$ so that

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2, \quad x \in \Omega, \quad \xi \in \mathbb{R}^n.$$

a3. The domain Ω is of class $C^{1,1}$.

a4. $a^{ij} \in W^{1,\infty}(\Omega)$, $1 \leq i, j \leq n$.

a5. $g \in W^{2-\frac{1}{p},p}(\Gamma)$ with $p > n$ and it is non identically equal to zero.

We further define \mathcal{Q}_0 as the subset of $L^\infty(\Omega)$ consisting of functions q such that 0 is not an eigenvalue of the realization of the operator L_q on $L^2(\Omega)$ under the Dirichlet boundary condition.

For $q \in \mathcal{Q}_0$, let $u_q \in H^1(\Omega)$ denotes the unique weak solution of the boundary value problem (abbreviated to BVP in the sequel)

$$(1.1) \quad \begin{cases} L_q u = 0 & \text{in } \Omega, \\ u = g & \text{on } \Gamma. \end{cases}$$

We will see in Section 3 that in fact $u_q \in W^{2,p}(\Omega)$, for any $q \in \mathcal{Q}_0$.

We assume in the present work that the matrix (a^{ij}) is known. We are then concerned with the inverse problem of reconstructing the coefficient $q = q(x)$ from the internal over-specified data

$$(1.2) \quad I_q(x) = q(x)u_q^2(x), \quad x \in \Omega.$$

In [Tr], the second author derived a Lipschitz stability estimate for the inverse problem above when the $(a^{ij}(x))$ is the identity matrix and when the Dirichlet boundary condition g is chosen in such a way that u_q does not vanish over Ω . It turns out that the Lipschitz constant is inversely proportional to $\min_\Omega |u_q|$ and can be very large if this later is close to zero.

We aim to establish stability estimates without such an assumption on the Dirichlet boundary condition g . Obviously, when u_q vanishes at a critical point \hat{x} , that is $u_q(\hat{x}) = 0$, we expect to lose informations on $q(x)$ in the surrounding area of the point \hat{x} . Henceforth, we predict that the stability estimate in such areas will deteriorate. How worse it can be ? Can we derive stability estimates that reflect our intuition ?

In fact, the presence of critical points where the collected internal data completely vanishes is actually a major problem in solving multi-waves inverse problems. The set of these critical points depend on the choice of the boundary conditions, and can be directly determined from the data itself [HN, Tr, BC]. In most existing stability results, the boundary conditions are assumed to be close to a set of CGO solutions (we refer to [SU] for more details on such solutions), where the critical points can be avoided [ACGRT, ABCTF, BU, BBMT, KS]. The goal of the present work is to avoid using such assumptions since they are not realistic in physical point of view.

The rest of this text is structured as follows. The main results are stated in Section 2. The well-posedness and the regularity of the solution to the direct problem are provided in Section 3. The uniqueness of the inverse problem is proved in Section 4. Weighted stability estimates for the electro-acoustic inverse problems

without assumptions on the boundary conditions are given in Section 5. Finally, we prove general logarithmic stability estimates in Section 6.

2. MAIN RESULTS

We state the main uniqueness and stability results of the paper.

Theorem 2.1. (Uniqueness) *We assume that conditions **a1-a5** are satisfied. Let $q, \tilde{q} \in \mathcal{Q}_0$ be such that*

$$\frac{\tilde{q}}{q} \in C(\overline{\Omega}) \quad \text{and} \quad \min_{\overline{\Omega}} \left| \frac{\tilde{q}}{q} \right| > 0,$$

Then $I_q = I_{\tilde{q}}$ implies $q = \tilde{q}$.

Next, we define the set of unknown coefficient for which we will prove a stability estimate. Fix $q_0 > 0$ and $0 < k < 1$. Let $q^* \in \mathcal{Q}_0$ such that $0 < 2q_0 \leq q^*$.

Given $q \in \mathcal{Q}_0$, we let A_q be the unbounded operator $A_q w = -L_q w$ with domain $D(A_q) = \{w \in H_0^1(\Omega); L_q w \in L^2(\Omega)\}$. The assumption that 0 is not an eigenvalue of A_q means that $A_q^{-1} : L^2(\Omega) \rightarrow L^2(\Omega)$ is bounded or, in other words, 0 belongs to the resolvent set of A_q .

We define \mathcal{Q} to be the set of those functions in $L^\infty(\Omega)$ satisfying

$$(2.1) \quad \|q - q^*\|_{L^\infty(\Omega)} \leq \min \left(\frac{k}{\|A_{q^*}^{-1}\|_{\mathcal{B}(L^2(\Omega))}}, q_0 \right).$$

We will see in Section 3 that $\mathcal{Q} \subset \mathcal{Q}_0$.

In the sequel C, C_j , where j is an integer, are generic constants that can only depend on $\Omega, (a^{ij}), n, g$, possibly on the a priori bounds we make on the unknown function q and, eventually, upon a fixed parameter $\theta \in (0, \frac{1}{4})$.

Theorem 2.2. (Weighted stability) *We assume that assumptions **a1-a5** are fulfilled. Let $q, \tilde{q} \in \mathcal{Q} \cap W^{1,\infty}(\Omega)$ satisfying $q - \tilde{q} \in H_0^1(\Omega)$ and*

$$\|q\|_{W^{1,\infty}(\Omega)}, \|\tilde{q}\|_{W^{1,\infty}(\Omega)} \leq M.$$

Then

$$(2.2) \quad \left\| \sqrt{I_q} (q - \tilde{q}) \right\|_{H^1(\Omega)} \leq C \left\| \sqrt{I_q} - \sqrt{I_{\tilde{q}}} \right\|_{H^1(\Omega)}^{\frac{1}{2}},$$

The result above shows that a Lipschitz stability holds far from the set of critical points where the solution vanishes.

Next, we derive general stability estimates without weight and which hold everywhere including in the vicinity of the critical points where u_q vanishes. The goal is to get rid of the weight $\sqrt{I_q}$ in the stability (2.2). The problem is reduced to solve the following linear inverse problem: reconstruct $f(x) \in L^\infty(\Omega)$ from the knowledge of $|u_q(x)|f(x)$, $x \in \Omega$. Obviously, if u_q does not vanish over Ω , the multiplication operator $M_{|u_q|}$ is invertible, and as in [Tr], a Lipschitz stability estimate will hold. Here, the zero set of u_q can be not empty. The main idea to overcome this difficulty consists in quantifying the unique continuation property of the operator L_q , that is, u_q can not vanish within a not empty open set (see for instance Lemmata (6.2) and (6.4)).

Prior to state our general stability estimate, we need to make additional assumptions:

a6. There exists a positive integer m such that any two disjoint points of Ω can be connected by a broken line consisting of at most m segments.

a7. $|g| > 0$ on Γ .

Remark 2.1. It is worthwhile to mention that assumption **a6** is equivalent to say Ω is multiply-starshaped. The notion of multiply-starshaped domain was introduced in [CJ] and it is defined as follows: the open subset D is said multiply-starshaped if there exists a finite number of points in D , say x_1, \dots, x_k , so that

(i) $\cup_{i=1}^{k-1} [x_i, x_{i+1}] \subset D$,

(ii) any point in D can be connected by a line segment to at least one of the points x_i .

In this case, any two points in D can be connected by a broken line consisting of at most $k + 1$ line segments. Obviously, the case $k = 1$ corresponds to the usual definition of a starshaped domain.

Henceforth, ϕ is a generic function of the form

$$\phi(s) = C_0 [|\ln C_1| |\ln(s)|^{-1} + s], \quad s > 0.$$

Theorem 2.3. (General stability) *We assume that conditions **a1-a7** hold. We fix $0 < \theta < \frac{1}{4}$ and $M > 0$. For any $q, \tilde{q} \in \mathcal{Q} \cap W^{1,\infty}(\Omega)$ satisfying $q - \tilde{q} \in H_0^1(\Omega)$ and*

$$\|q\|_{W^{1,\infty}(\Omega)}, \|\tilde{q}\|_{W^{1,\infty}(\Omega)} \leq M.$$

it holds that

$$\|q - \tilde{q}\|_{L^\infty(\Omega)} \leq \phi \left(\left\| \sqrt{I_q} - \sqrt{I_{\tilde{q}}} \right\|_{H^1(\Omega)}^\theta \right).$$

We still have a weak version of the stability estimate above even if we do not assume that condition **a7** holds. Namely, we prove

Theorem 2.4. (General weak stability) *Let assumptions **a1-a6** hold. We fix $0 < \theta < \frac{1}{4}$, $M > 0$ and ω an open neighborhood of Γ in $\bar{\Omega}$. For all $q, \tilde{q} \in \mathcal{Q} \cap W^{1,\infty}(\Omega)$ satisfying $q = \tilde{q}$ on $\omega \cup \Gamma$ and*

$$\|q\|_{W^{1,\infty}(\Omega)}, \|\tilde{q}\|_{W^{1,\infty}(\Omega)} \leq M.$$

it holds that

$$\|q - \tilde{q}\|_{L^\infty(\Omega)} \leq \phi \left(\left\| \sqrt{I_q} - \sqrt{I_{\tilde{q}}} \right\|_{H^1(\Omega)}^\theta \right).$$

The general stability estimates indicate that the inverse problem of recovering the coefficient q in the vicinity of the critical points is severely ill-posed. The key point in proving our general stability results consists in a estimating the local behaviour of the solution of the BVP (1.1) in terms of L^2 norm. To do so, we adapt the method introduced in [BCJ] to derive a stability estimate for the problem of recovering the surface impedance of an obstacle from the the far field pattern. This method was also used in [CJ] to establish a stability estimate for the problem detecting a corrosion from a boundary measurement.

3. $W^{2,p}$ -REGULARITY OF THE SOLUTION OF THE BVP

In all of this section, we assume that assumptions **a1-a5** are fulfilled.

Theorem 3.1. *i) Let $q \in \mathcal{Q}_0$. Then u_q , the unique weak solution of the BVP (1.1), belongs to $W^{2,p}(\Omega)$.
ii) We have the following a priori bound*

$$(3.1) \quad \|u_q\|_{W^{2,p}(\Omega)} \leq C, \quad q \in \mathcal{Q}.$$

Proof. i) Let $q \in \mathcal{Q}_0$. We pick $G \in W^{2,p}(\Omega)$ such that $G = g$ on Γ and we set $f_q = L_q G$. From our assumptions on a^{ij} , $f_q \in L^p(\Omega)$ and therefore $v_q = u_q - G$ is the weak solution of the following BVP

$$\begin{cases} -L_q v = f_q & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

That is $v_q = A_q^{-1} f_q$. In light of the fact that $W^{2,p}(\Omega)$ is continuously embedded in $H^2(\Omega)$, we get from the classical H^2 -regularity theorem (e.g. [RR][Theorem 8.53, page 326]), that $u_q = G + A_q^{-1} f_q \in H^2(\Omega)$.

But, $H^2(\Omega)$ is continuously embedded in $L^\infty(\Omega)$ when $n = 2$ or $n = 3$. Therefore,

$$-\partial_i(a^{ij}\partial_j u_q) = q u_q \in L^\infty(\Omega).$$

Hence, $u_q \in W^{2,p}(\Omega)$ by [GT][Theorem 9.15, page 241].

ii) Let $q \in \mathcal{Q}$. If M_{q^*-q} is the multiplier by $q^* - q$, acting on $L^2(\Omega)$, then $u_q - u_{q^*} \in D(A_q)$ and

$$(3.2) \quad A_q(u_q - u_{q^*}) = M_{q^*-q} u_{q^*}.$$

On the other hand, we have

$$A_q = A_{q^*} (I + A_{q^*}^{-1} M_{q^*-q}).$$

Since by our assumption

$$\|A_{q^*}^{-1}\|_{\mathcal{B}(L^2(\Omega))} \|q - q^*\|_{L^\infty(\Omega)} \leq k < 1,$$

the operator $I + A_{q^*}^{-1} M_{q^*-q}$ is an isomorphism on $L^2(\Omega)$ and

$$\|(I + A_{q^*}^{-1} M_{q^*-q})^{-1}\|_{\mathcal{B}(L^2(\Omega))} \leq \frac{1}{1-k}.$$

Hence, A_q^{-1} is bounded and

$$\|A_q^{-1}\|_{\mathcal{B}(L^2(\Omega))} = \|(I + A_{q^*}^{-1} M_{q^*-q})^{-1} A_{q^*}^{-1}\|_{\mathcal{B}(L^2(\Omega))} \leq \frac{\|A_{q^*}^{-1}\|_{\mathcal{B}(L^2(\Omega))}}{1-k}.$$

In light of (3.2), we get

$$\|u_q\|_{L^2(\Omega)} \leq \|u_{q^*}\|_{L^2(\Omega)} + \|u_q - u_{q^*}\|_{L^2(\Omega)} \leq \|u_{q^*}\|_{L^2(\Omega)} \left(1 + \frac{k}{k-1}\right).$$

In combination with [RR] [(8.190), page 326], this estimate implies $\|u_q\|_{H^2(\Omega)} \leq C$ and therefore, bearing in mind that $H^2(\Omega)$ is continuously embedded in $L^\infty(\Omega)$,

$$\|u_q\|_{L^\infty(\Omega)} \leq C$$

Finally, from [GT] [Lemma 9.17, page 242], we find

$$\|u_q\|_{W^{2,p}(\Omega)} \leq C_1 \|qu_q\|_{L^p(\Omega)} + \|L_0 G\|_{L^p(\Omega)} \leq C.$$

□

Taking into account that $W^{2,p}(\Omega)$ is continuously embedded into $C^{1,\beta}(\overline{\Omega})$ with $\beta = 1 - \frac{n}{p}$, we obtain as straightforward consequence of estimate (3.1) the following corollary:

Corollary 3.1.

$$(3.3) \quad \|u_q\|_{C^{1,\beta}(\overline{\Omega})} \leq C, \quad q \in \mathcal{Q}.$$

Remark 3.1. Estimate (3.3) is crucial in establishing our general uniform stability estimates. Specifically, estimate (3.3) is necessary to get that ϕ in Theorems 2.3 and 2.4 doesn't depend on q and \tilde{q} . Also, the $C^{1,\beta}$ -regularity of u_q guaranties that $|u_q|$ is Lipschitz continuous which is a key point in the proof of the weighted stability estimate.

4. UNIQUENESS

In this section, even if it is not necessary, we assume for simplicity that conditions **a1-a5** hold true.

The following Caccioppoli's inequality will be useful in the sequel. These kind of inequalities are well known (e.g. [Mo]), but for sake of completeness we give its (short) proof.

Lemma 4.1. *Let $q \in \mathcal{Q}_0$ satisfying $\|q\|_{L^\infty(\Omega)} \leq \Lambda$, for a given $\Lambda > 0$. Then there exists a constant $\widehat{C} = \widehat{C}(\Omega, (a^{ij}), \Lambda) > 0$ such that, for any $x \in \Omega$ and $0 < r < \frac{1}{2} \text{dist}(x, \Gamma)$,*

$$(4.1) \quad \int_{B(x,r)} |\nabla u_q|^2 dy \leq \frac{\widehat{C}}{r^2} \int_{B(x,2r)} u_q^2 dy.$$

Proof. We start by noticing that the following identity holds true in a straightforward way

$$(4.2) \quad \int_{\Omega} a^{ij} \partial_i u_q \partial_j v dy = \int_{\Omega} qu_q v dy, \quad v \in C_0^1(\Omega),$$

We pick $\chi \in C_c^\infty(B(x, 2r))$ satisfying $0 \leq \chi \leq 1$, $\chi = 1$ in a neighborhood of $B(x, r)$ and $|\partial^\gamma \chi| \leq Kr^{-|\gamma|}$ for $|\gamma| \leq 2$, where K is a constant not depending on r . Then identity (4.2) with $v = \chi u$ gives

$$\begin{aligned} \int_{\Omega} \chi a^{ij} \partial_i u_q \partial_j u_q dy &= - \int_{\Omega} u_q a^{ij} \partial_i u \partial_j \chi dy + \int_{\Omega} \chi q u_q^2 dy \\ &= -\frac{1}{2} \int_{\Omega} a^{ij} \partial_i u_q^2 \partial_j \chi dy + \int_{\Omega} \chi q u_q^2 dy \\ &= \frac{1}{2} \int_{\Omega} u_q^2 \partial_i (a^{ij} \partial_j \chi) dy + \int_{\Omega} \chi q u_q^2 dy. \end{aligned}$$

Therefore, since

$$\int_{\Omega} \chi a^{ij} \partial_i u_q \partial_j u_q dy \geq \lambda \int_{\Omega} \chi |\nabla u_q|^2 dy,$$

(4.1) follows immediately. \square

Theorem 4.1. *For any compact subset $K \subset \Omega$, $u_q^{-2r} \in L^1(K)$, for some $r = r(K, u_q) > 0$.*

Proof. For sake of simplicity, we use in this proof u in place of u_q .

We first prove that u^2 is locally a Muckenhoupt weight. We follow the method introduced by Garofalo and Lin in [GL].

Let $B_{4R} = B(x, 4R) \subset \Omega$. According to Proposition 3.1 in [MV] (page 7), we have the following so-called doubling inequality

$$(4.3) \quad \int_{B_{2r}} u^2 dy \leq \tilde{C} \int_{B_r} u^2 dy, \quad 0 < r \leq R.$$

Here and until the end of the proof, \tilde{C} denotes a generic constant that can depend on u and R but not in r .

Since $\partial_j(a^{ij} \partial_i u^2) = 2u \partial_j(a^{ij} \partial_i u) + 2a^{ij} \partial_i u \partial_j u$, we have

$$\partial_j(a^{ij} \partial_i u^2) + 2qu^2 = 2a^{ij} \partial_i u \partial_j u \geq \lambda |\nabla u|^2 \geq 0 \text{ in } \Omega.$$

By [GT][Theorem 9.20, page 244], we have, noting that $u^2 \in W^{2,n}(B_{4R})$,

$$(4.4) \quad \sup_{B_r} u^2 \leq \frac{\tilde{C}}{|B_{2r}|} \int_{B_{2r}} u^2 dy.$$

On the other hand, we have trivially

$$(4.5) \quad \left(\frac{1}{|B_r|} \int_{B_r} u^{2(1+\delta)} dx \right)^{\frac{1}{1+\delta}} \leq \sup_{B_r} u^2, \text{ for any } \delta > 0.$$

From (4.3), (4.4) and (4.5), we obtain

$$(4.6) \quad \left(\frac{1}{|B_r|} \int_{B_r} u^{2(1+\delta)} dy \right)^{\frac{1}{1+\delta}} \leq \tilde{C} \left(\frac{1}{|B_r|} \int_{B_r} u^2 dy \right), \text{ for any } \delta > 0.$$

In other words, u^2 satisfies a reverse Hölder inequality. Therefore, we can apply a theorem by Coifman and Fefferman [CF] (see also [Ko][Theorem 5.42, page 136]). We get

$$(4.7) \quad \frac{1}{|B_r|} \int_{B_r} u^2 dy \left(\frac{1}{|B_r|} \int_{B_r} u^{-\frac{2}{\kappa-1}} dy \right)^{\kappa-1} \leq \tilde{C},$$

where $\kappa > 1$ is a constant depending on u but not in r .

By the usual unique continuation property u cannot vanish identically on $B_R = B_R(x)$. Therefore (4.3) implies

$$(4.8) \quad \int_{B_R(x)} u^{-2r(x)} dy \leq \tilde{C}.$$

Let K be a compact subset of Ω . Then K can be covered by a finite number, say N , of balls $B_i = B_R(x_i)$. Let $(\varphi_i)_{1 \leq i \leq N}$ be a partition of unity subordinate to the covering (B_i) . If $r = r(K) = \min\{r(x_i); 1 \leq i \leq N\}$, then

$$\int_K u^{-2r} dy = \sum_{i=1}^N \int_K u^{-2r} \varphi_i dy \leq \sum_{i=1}^N \|\varphi_i\|_\infty \int_{B_i} u^{-2r} dy.$$

□

Proof of Theorem 2.1. As before, for simplicity, we use u (resp. \tilde{u}) in place of u_q (resp. $u_{\tilde{q}}$).

Let (Ω_k) be an increasing sequence of open subsets of Ω , $\Omega_k \subseteq \Omega$ for each k and $\cup_k \Omega_k = \Omega$. By Theorem 4.1, for each k ,

$$\frac{u^2}{\tilde{u}^2} = \frac{\tilde{q}}{q} \quad \text{a.e. in } \Omega_k$$

Therefore,

$$\frac{u^2}{\tilde{u}^2} = \frac{\tilde{q}}{q} \quad \text{a.e. in } \Omega.$$

Changing $\frac{u^2}{\tilde{u}^2}$ by its continuous representative $\frac{\tilde{q}}{q}$, we can assume that $\frac{u^2}{\tilde{u}^2} \in C(\overline{\Omega})$. Moreover, the last identity implies also that $\frac{\tilde{q}}{q} \geq 0$. Or $\min_{\overline{\Omega}} \left| \frac{\tilde{q}}{q} \right| > 0$. Therefore, $\frac{u}{\tilde{u}}$ is of constant sign and doesn't vanish. But, $\frac{u}{\tilde{u}} = 1$ on $\{x \in \Gamma; g(x) \neq 0\}$. Hence,

$$(4.9) \quad \frac{u}{\tilde{u}} = \sqrt{\frac{\tilde{q}}{q}}.$$

Let $v = u - \tilde{u}$. Using $I_q = I_{\tilde{q}}$ and (4.9), we obtain by a straightforward computation that v is a solution of the BVP

$$\begin{cases} -\partial_i(a^{ij}\partial_j v) + \sqrt{\tilde{q}q}v = 0 & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma. \end{cases}$$

Since the operator $-\partial_i(a^{ij}\partial_j \cdot) + \sqrt{\tilde{q}q}$ under Dirichlet boundary condition is strictly elliptic, we conclude that $v = 0$. Hence, $u = \tilde{u}$ and consequently $q = \tilde{q}$. □

5. WEIGHTED STABILITY ESTIMATES

We show that $|u_q|$, $q \in \mathcal{Q}_0$, is a solution of a certain BVP. To do so, we first prove that $|u_q|$ is Lipschitz continuous.

Proposition 5.1. *We have $|u_q| \in C^{0,1}(\overline{\Omega})$, for any $q \in \mathcal{Q}_0$.*

Proof. Let $q \in \mathcal{Q}_0$. We recall that, since $u_q \in H^1(\Omega)$, $|u_q| \in H^1(\Omega)$ and

$$(5.1) \quad \partial_i |u_q| = \text{sg}_0(u) \partial_i u_q \quad \text{a.e. in } \Omega.$$

Here

$$\text{sg}_0(s) = \begin{cases} 1 & s > 0, \\ 0 & s = 0 \\ -1 & s < 0 \end{cases}$$

We saw in the preceding section that $u \in C^{1,\beta}(\overline{\Omega})$, $\beta = 1 - \frac{n}{p}$. Hence, $u \in C^{0,1}(\overline{\Omega})$ because $C^{1,\beta}(\overline{\Omega})$ is continuously embedded in $C^{0,1}(\overline{\Omega})$. Then $|u_q| \in C^{0,1}(\overline{\Omega})$ as a straightforward consequence of the elementary inequality $||s| - |t|| \leq |s - t|$, for any scalar numbers s and t . □

Let $q \in \mathcal{Q}$. For simplicity, we use in the sequel u instead of u_q . It follows from Proposition (5.1) that $|u|^2$ lies in $H^1(\Omega)$ and satisfies

$$\partial_i |u|^2 = 2|u| \partial_i |u| \quad \text{a.e. in } \Omega.$$

Obviously, $\partial_i u^2 = \partial_i |u|^2$, and so

$$(5.2) \quad |u| \partial_i |u| = u \partial_i u \quad \text{a.e. in } \Omega.$$

A straightforward computation gives

$$\partial_i (a^{ij} \partial_j u^2) - 2a^{ij} \partial_i u \partial_j u = -2qu^2 \quad \text{a.e. in } \Omega.$$

Using identity (5.1), we can rewrite the equation above as follows

$$(5.3) \quad \partial_i (a^{ij} \partial_j |u|^2) - 2a^{ij} \partial_i |u| \partial_j |u| = -2qu^2 \quad \text{a.e. in } \Omega.$$

Using the fact that $|u|H_0^1(\Omega) \subset H_0^1(\Omega)$ and integrations by parts, we get

$$|u| \partial_i (a^{ij} \partial_j |u|) = \partial_i (a^{ij} |u| \partial_j |u|) - a^{ij} \partial_j |u| \partial_i |u| \quad \text{in } H^{-1}(\Omega).$$

But the right hand side of this identity belongs to $L^2(\Omega)$. Therefore,

$$(5.4) \quad \partial_i (a^{ij} |u| \partial_j |u|) = |u| \partial_i (a^{ij} \partial_j |u|) + a^{ij} \partial_j |u| \partial_i |u| \quad \text{a.e. in } \Omega.$$

Bearing in mind identities (5.4) and (5.2), we obtain by a simple calculation

$$(5.5) \quad \partial_i (a^{ij} \partial_j |u|^2) = 2a^{ij} \partial_i |u| \partial_j |u| + 2|u| \partial_i (a^{ij} \partial_j |u|) \quad \text{a.e. in } \Omega,$$

which, combined with (5.3), implies

$$(5.6) \quad |u| \partial_i (a^{ij} \partial_j |u|) = -qu^2 \quad \text{a.e. in } \Omega.$$

Now, as $q \geq q_0 > 0$ in Ω , $\mu = \frac{1}{\sqrt{q}}$ is well defined. We set $J = J_q = \sqrt{I_q}$. Then substituting $|u|$ by $J\mu$ in (5.6), we get

$$(5.7) \quad J \partial_i (a^{ij} \partial_j (J\mu)) = -\frac{J^2}{\mu} \quad \text{a.e. in } \Omega.$$

Proof of Theorem 2.2. Let $\tilde{\mu} = \frac{1}{\sqrt{\tilde{q}}}$ and $\tilde{J} = \sqrt{\tilde{I}_{\tilde{q}}}$. Identity (5.7) with q substituted by \tilde{q} yields

$$(5.8) \quad \tilde{J} \partial_i (a^{ij} \partial_j (\tilde{J}\tilde{\mu})) = -\frac{\tilde{J}^2}{\tilde{\mu}} \quad \text{a.e. in } \Omega.$$

Taking the difference of each side of equations (5.7) and (5.8), we obtain

$$(5.9) \quad J \partial_i (a^{ij} \partial_j (J(\mu - \tilde{\mu}))) - \frac{J^2}{\mu\tilde{\mu}} (\mu - \tilde{\mu}) dx = \frac{\tilde{J}^2 - J^2}{\tilde{\mu}} \\ + J \partial_i (a^{ij} \partial_j (\tilde{\mu}(\tilde{J} - J))) + (\tilde{J} - J) \partial_i (a^{ij} \partial_j (\tilde{\mu}\tilde{J})) \quad \text{a.e. in } \Omega.$$

We multiply each side of the identity above by $\mu - \tilde{\mu}$ and we integrate over Ω . We apply then the Green's formula to the resulting identity. Taking into account that $J = \tilde{J}$ and $\mu = \tilde{\mu}$ on Γ , we obtain

$$\int_{\Omega} a^{ij} \partial_i (J(\mu - \tilde{\mu})) \partial_j (J(\mu - \tilde{\mu})) dx + \int_{\Omega} \frac{1}{\mu\tilde{\mu}} J^2 (\mu - \tilde{\mu})^2 dx = \int_{\Omega} \frac{1}{\tilde{\mu}} (J^2 - \tilde{J}^2) (\mu - \tilde{\mu}) dx \\ + \int_{\Omega} a^{ij} \partial_i (\tilde{\mu}(J - \tilde{J})) \partial_j (J(\mu - \tilde{\mu})) dx \\ + \int_{\Omega} a^{ij} \partial_i (\tilde{\mu}\tilde{J}) \partial_j ((\tilde{J} - J)(\mu - \tilde{\mu})) dx.$$

We apply the Cauchy-Schwarz's inequality to each term of the right hand side of the above identity. We obtain

$$\tilde{C}_0 \|J(\mu - \tilde{\mu})\|_{H^1(\Omega)}^2 \leq \tilde{C}_1 \left(\|J\|_{L^2(\Omega)} + \|\tilde{J}\|_{L^2(\Omega)} \right) \|J - \tilde{J}\|_{H^1(\Omega)} + \tilde{C}_2 \|J\|_{H^1(\Omega)} \|J - \tilde{J}\|_{H^1(\Omega)},$$

where

$$\begin{aligned}\tilde{C}_0 &= \min(\lambda, q_0), \\ \tilde{C}_1 &= \left(1 + \left(\frac{M}{q_0}\right)^{\frac{1}{2}}\right) + \|a^{ij}\|_{L^\infty(\Omega)} \left(\frac{1}{\sqrt{q_0}} + \frac{M}{q_0}\right), \\ \tilde{C}_2 &= \|a^{ij}\|_{L^\infty(\Omega)} \left(\frac{1}{\sqrt{q_0}} + \frac{M}{q_0}\right)^2.\end{aligned}$$

Clearly, this estimate leads to (2.2). \square

6. GENERAL STABILITY ESTIMATES

We assume in the present section, even if it is not necessary, that conditions **a1-a6** are satisfied.

6.1. Local behaviour of the solution of the BVP. We pick $q \in \mathcal{Q}_0$ and we observe that our assumption on Ω implies that this later possesses the uniform interior cone property. That is, there exist $R > 0$ and $\theta \in]0, \frac{\pi}{2}[$ such that for all $\tilde{x} \in \Gamma$, we find $\xi \in \mathbb{S}^{n-1}$ for which

$$\mathcal{C}(\tilde{x}) = \{x \in \mathbb{R}^n; |x - \tilde{x}| < 2R, (x - \tilde{x}) \cdot \xi > |x - \tilde{x}| \cos \theta\} \subset \Omega.$$

We set $x_0 = \tilde{x} + R\xi$ and, for $0 < r < R\left(\frac{3}{\sin \theta} + 1\right)^{-1}$, and we consider the sequence

$$x_k = \tilde{x} + (R - kr)\xi, \quad k \geq 1.$$

Let N_0 be the smallest integer satisfying

$$(R - (N_0 + 1)r) \sin \theta \leq 3r.$$

This and the fact that $(R - N_0 r) \sin \theta > 3r$ imply

$$(6.1) \quad \frac{R}{r} - \frac{3}{\sin \theta} - 1 \leq N_0 \leq \frac{R}{r} - \frac{3}{\sin \theta}.$$

Let assumption **a7** holds and set $2\eta = \min_\Gamma |g| > 0$. In light of estimate (3.3), we get $|u_q| \geq \eta$ in a neighborhood (independent on q) of Γ in $\bar{\Omega}$. Or

$$|x_N - \tilde{x}| = R - N_0 r \leq \left(\frac{3}{\sin \theta} + 1\right) r.$$

Therefore, there exists r^* such that $|u_q| \geq \eta$ in $B(x_{N_0}, r)$, for all $0 < r \leq r^*$. Consequently,

$$(6.2) \quad Cr^{\frac{n}{2}} \leq \|u_q\|_{L^2(B(x_{N_0}, r))}.$$

We shall use the following three spheres Lemma. In the sequel, we assume $\|q\|_{L^\infty(\Omega)} \leq \Lambda$.

Lemma 6.1. *There exist $N > 0$ and $0 < s < 1$, that only depend on Ω , (a^{ij}) and $\Lambda > 0$ such that, for all $w \in H^1(\Omega)$ satisfying $L_q w = 0$ in Ω , $y \in \Omega$ and $0 < r < \frac{1}{3} \text{dist}(y, \Gamma)$,*

$$r\|w\|_{H^1(B(y, 2r))} \leq N\|w\|_{H^1(B(y, r))}^s \|w\|_{H^1(B(y, 3r))}^{1-s}.$$

For sake of completeness, we prove this lemma in appendix A by means of a Carleman inequality.

We introduce the following temporary notation

$$I_k = \|u_q\|_{H^1(B(x_k, r))}, \quad 0 \leq k \leq N_0.$$

Since $B(x_k, r) \subset B(x_{k-1}, 2r)$, $1 \leq k \leq N_0$, we can apply Lemma 6.1. By an induction argument in k , we obtain

$$I_{N_0} \leq \left(\frac{C}{r}\right)^{\frac{1-sN_0}{1-s}} C^{1-sN_0} I_0^{sN_0},$$

Shortening if necessary r^* , we can always assume that $\frac{C}{r^*} \geq 1$. Hence

$$(6.3) \quad I_{N_0} \leq \frac{C}{r^t} I_0^{sN_0}, \quad t = \frac{1}{1-s}.$$

A combination of (6.2) and (6.3) yields

$$Cr^{\frac{n}{2}+t} \leq I_0^{s^{N_0}}.$$

That is

$$(6.4) \quad Cr^{\frac{n}{2}+t} \leq \|u\|_{H^1(B(x_0, r))}^{s^{N_0}}.$$

Now, let $y_0 \in \Omega$ such that the segment $[x_0, y_0]$ lies entirely in Ω . Set

$$d = |x_0 - y_0| \quad \text{and} \quad \zeta = \frac{y_0 - x_0}{|x_0 - y_0|}.$$

Consider the sequence, where $0 < 2r < d$,

$$y_k = y_0 - k(2r)\zeta, \quad k \geq 1.$$

We have

$$|y_k - x_0| = d - k(2r).$$

Let N_1 be the smallest integer such that $d - N_1(2r) \leq r$, or equivalently

$$\frac{d}{2r} - \frac{1}{2} \leq N_1 < \frac{d}{2r} + \frac{1}{2}.$$

Using again Lemma 6.1, we obtain

$$(6.5) \quad \tilde{C}r^t \|u\|_{H^1(B(y_{N_1}, 2r))} \leq \|u\|_{H^1(B(y_0, 2r))}^{s^{N_1}}.$$

Since $|y_{N_1} - x_0| = d - N_1(2r) \leq r$, $B(x_0, r) \subset B(y_{N_1}, 2r)$. From (6.4) and (6.5), shortening again r^* if necessary, we have

$$\tilde{C}r^t (Cr^{\frac{n}{2}+t})^{\frac{1}{s^{N_0}}} \leq \|u\|_{H^1(B(y_0, 2r))}^{s^{N_1}}.$$

Reducing again r^* if necessary, we can assume that $\tilde{C}r^t < 1$, $0 < r \leq r^*$. Therefore, we find

$$(Cr)^{\frac{\frac{n}{2}+2t}{s^{N_0}+N_1}} \leq \|u\|_{H^1(B(y_0, 2r))}.$$

Let us observe that we can repeat the argument between x_0 and y_0 to y_0 and $z_0 \in \Omega$ such that $[y_0, z_0] \subset \Omega$ and so on. By this way and since we have assumed that each two points of Ω can be connected by a broken line consisting of at most m segments, we have the following result.

Lemma 6.2. *Let assumption **a7** be satisfied. There are constants $c > 0$ and $r^* > 0$, that can only depend on data and Λ , such that for any $x \in \bar{\Omega}$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u_q\|_{H^1(B(x, r) \cap \Omega)}, \quad 0 < r \leq r^*.$$

Remark 6.1. *We observe that we implicitly use the following property for establishing Lemma 6.2, which is a direct consequence of the regularity of Ω : if $\tilde{x} \in \Gamma$, then $|B(\tilde{x}, r) \cap \Omega| \geq \kappa |B(\tilde{x}, r)|$, $0 < r$, where the constant $\kappa \in (0, 1)$ does not depend on \tilde{x} and r .*

Observing again that $|u_q| \geq \eta$ in a neighborhood of Γ in $\bar{\Omega}$, with $\eta > 0$ is as above, a combination of Lemma 6.2 and Lemma 4.1 (Caccioppoli's inequality) leads to

Corollary 6.1. *Under assumption **a7**, there are constants $c > 0$ and $r^* > 0$, that can depend only on data and Λ , such that for any $x \in \bar{\Omega}$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u_q\|_{L^2(B(x, r) \cap \Omega)}, \quad 0 < r \leq r^*.$$

From this corollary, we get the key lemma that we will use to establish general stability estimates for our inverse problem.

Lemma 6.3. *Assume that **a7** is fulfilled. There exists δ^* so that, for any $x \in \Omega$ and $0 < \delta \leq \delta^*$,*

$$\{y \in B(x, r^*) \cap \Omega; u_q(x)^2 \geq \delta\} \neq \emptyset.$$

Proof. If the result does not hold we can find a sequence (δ_k) , $0 < \delta_k \leq \frac{1}{k}$ such that

$$\{y \in B(x, r^*) \cap \Omega; u_q(x)^2 \geq \delta_k\} = \emptyset.$$

Then

$$u_q^2 \leq \frac{1}{k} \text{ in } B(x, r^*) \cap \Omega.$$

In light of Corollary 6.1, we obtain

$$e^{-ce^{\frac{c}{r^*}}} \leq \frac{1}{\sqrt{k}} |B(x, r^*) \cap \Omega| \leq \frac{|\Omega|}{\sqrt{k}}, \text{ for all } k \geq 1.$$

This is impossible. Hence, the desired contradiction follows. \square

Further, we recall that the semi-norm $[f]_\alpha$, for $f \in C^\alpha(\overline{\Omega})$, is given as follows:

$$[f]_\alpha = \sup\{|f(x) - f(y)| |x - y|^{-\alpha}; x, y \in \overline{\Omega}, x \neq y\}.$$

Proposition 6.1. *Let $M > 0$ be given and assume that assumption **a7** holds. Then, for any $q \in \mathcal{Q}$, $f \in C^\alpha(\overline{\Omega})$ satisfying $\|f\|_{C^\alpha(\overline{\Omega})} \leq M$, we have*

$$\|f\|_{L^\infty(\Omega)} \leq \phi(\|fu_q^2\|_{L^\infty(\Omega)}).$$

Proof. Let δ^* be as in Corollary 6.1, $f \in C^\alpha(\overline{\Omega})$ and $0 < \delta \leq \delta^*$. For $x \in \overline{\Omega}$, we consider two cases: (a) $u_q(x)^2 \geq \delta$ and (b) $u_q(x)^2 \leq \delta$.

(a) $u_q(x)^2 \geq \delta$. We have

$$(6.6) \quad |f(x)| \leq \frac{1}{\delta} |f(x)u_q(x)^2|.$$

(b) $u_q(x)^2 \leq \delta$. We set

$$r = \sup\{0 < \rho; u_q^2 < \delta \text{ on } B(x, \rho) \cap \overline{\Omega}\}.$$

By Lemma 6.3

$$\{x \in B(x, r^*) \cap \overline{\Omega}; u_q(x)^2 \geq \delta\} \neq \emptyset.$$

Hence, $r \leq r^*$ and

$$\partial B(x, r) \cap \{x \in B(x, r^*) \cap \overline{\Omega}; u_q(x)^2 \geq \delta\} \neq \emptyset.$$

Let $y \in \partial B(x, r)$ be such that $u_q(y)^2 \geq \delta$. We have,

$$|f(x)| \leq |f(x) - f(y)| + |f(y)| \leq [f]_\alpha |x - y|^\alpha + \frac{1}{\delta} |f(y)u_q(y)^2|$$

and then

$$|f(x)| \leq Mr^\alpha + \frac{1}{\delta} |f(y)u_q(y)^2|.$$

This and (6.6) show

$$(6.7) \quad \|f\|_{L^\infty(\Omega)} \leq Mr^\alpha + \frac{1}{\delta} \|fu_q^2\|_{L^\infty(\Omega)}.$$

Since $u_q^2 \leq \delta$ in $B(x, r) \cap \overline{\Omega}$, Corollary 6.1 implies

$$e^{-ce^{\frac{c}{r}}} \leq \sqrt{\delta} |B(x, r) \cap \Omega| \leq \sqrt{\delta} |\Omega|,$$

or equivalently

$$r \leq \frac{c}{\ln\left(\frac{1}{c} |\ln |\Omega| \sqrt{\delta}| \right)}.$$

This estimate in (6.7) yields

$$\|f\|_{L^\infty(\Omega)} \leq \frac{C_0}{\left[\ln\left(C_1 |\ln |\Omega| \sqrt{\delta}| \right)\right]^\alpha} + \frac{1}{\delta} \|fu_q^2\|_{L^\infty(\Omega)}.$$

Setting $e^s = C_1 |\ln |\Omega| \sqrt{\delta}|$, the last inequality can be rewritten as

$$\|f\|_{L^\infty(\Omega)} \leq \frac{C_0}{s^\alpha} + |\Omega| e^{\frac{2}{C_1} e^s} \|f u_q^2\|_{L^\infty(\Omega)}.$$

An usual minimization argument with respect to s leads to the existence of $\epsilon > 0$, depending only on data, so that

$$(6.8) \quad \|f\|_{L^\infty(\Omega)} \leq \psi(\|f u_q^2\|_{L^\infty(\Omega)}),$$

with ψ a function of the form

$$\psi(s) = C_2 |\ln C_3| |\ln(s)|^{-1/\alpha}$$

provided that $\|f u_q^2\|_{L^\infty(\Omega)} \leq \epsilon$.

When $\|f u_q^2\|_{L^\infty(\Omega)} \geq \epsilon$, we have trivially

$$(6.9) \quad \|f\|_{L^\infty(\Omega)} \leq \frac{M}{\epsilon} \|f u_q^2\|_{L^\infty(\Omega)}.$$

The desired estimate follows by combining (6.8) and (6.9). \square

We now turn our attention to the case without assumption **a7**. First, we observe that we still have a weaker version of Corollary 6.1. In fact all our analysis leading to Lemma 6.2 holds whenever we start with $\tilde{x} \in \Gamma$ satisfying $2\eta = |g(\tilde{x})| \neq 0$ and take x_0 in a neighborhood of \tilde{x} such that $|u| \geq \eta$ in a ball around x_0 (we note that, as we have seen before, such a ball can be chosen independently on q).

Lemma 6.4. *Let K be a compact subset of Ω . There exist $r^* > 0$ and $c > 0$, depending only on data, Λ and K , so that, for any $x \in K$,*

$$e^{-ce^{\frac{c}{r}}} \leq \|u\|_{H^1(B(x,r))}, \quad 0 < r \leq r^*.$$

In light of this lemma, an adaptation of the proofs of Lemma 6.3 and Proposition 6.1 yields

Proposition 6.2. *Given $M > 0$ and K a compact subset of Ω , for any $f \in C^\alpha(\overline{\Omega})$ satisfying $\|f\|_{C^\alpha(\overline{\Omega})} \leq M$, we have*

$$\|f\|_{L^\infty(K)} \leq \phi(\|f u^2\|_{L^\infty(\Omega)}).$$

6.2. Proof of stability estimates.

Proof of Theorem 2.3. Set $U = u_q^2$ and $\tilde{U} = u_q^2$. Then a straightforward computation shows

$$\sqrt{U} - \sqrt{\tilde{U}} = \sqrt{I} \left(\frac{1}{\sqrt{q}} - \frac{1}{\sqrt{\tilde{q}}} \right) + \frac{1}{\sqrt{\tilde{q}}} (\sqrt{I} - \sqrt{\tilde{I}}).$$

This identity, in combination with Theorem 2.2, gives

$$(6.10) \quad \begin{aligned} \|U - \tilde{U}\|_{H^1(\Omega)} &\leq \left(\|\sqrt{U}\|_{H^1(\Omega)} + \|\sqrt{\tilde{U}}\|_{H^1(\Omega)} \right) \|\sqrt{U} - \sqrt{\tilde{U}}\|_{H^1(\Omega)} \\ &\leq C \left(\|\sqrt{U}\|_{H^1(\Omega)} + \|\sqrt{\tilde{U}}\|_{H^1(\Omega)} \right) \left(1 + \|\sqrt{U}\|_{H^1(\Omega)} \right) \|\sqrt{I} - \sqrt{\tilde{I}}\|_{H^1(\Omega)}^{\frac{1}{2}}. \end{aligned}$$

On the other hand, since $L^\infty(\Omega)$ is continuously embedded in $H^{2-2\theta}$,

$$\|U - \tilde{U}\|_{L^\infty(\Omega)} \leq C \|U - \tilde{U}\|_{H^{2-2\theta}(\Omega)}.$$

This, the interpolation inequality (e.g. [LM][Remark 9.1, page 49])

$$\|w\|_{H^{2-2\theta}(\Omega)} \leq C \|w\|_{H^1(\Omega)}^{2\theta} \|w\|_{H^2(\Omega)}^{1-2\theta}, \quad w \in H^2(\Omega),$$

and estimate (3.1) imply

$$(6.11) \quad \|U - \tilde{U}\|_{L^\infty(\Omega)} \leq C \|\sqrt{I} - \sqrt{\tilde{I}}\|_{H^1(\Omega)}^\theta.$$

But

$$(q - \tilde{q}) u^2 = I - \tilde{I} + \tilde{q} (U - \tilde{U}).$$

Then (6.11) yields

$$\|(q - \tilde{q}) u^2\|_{L^\infty(\Omega)} \leq C \left\| \sqrt{I} - \sqrt{\tilde{I}} \right\|_{H^1(\Omega)}^\theta.$$

We complete the proof by applying Proposition 6.1 with $f = q - \tilde{q}$. \square

Proof of Theorem 2.4. Similar to the previous one. We have only to apply Proposition 6.2 instead of Proposition 6.1. \square

7. CONCLUSION

This paper has investigated the inverse problem of recovering a coefficient of a Helmholtz operator from internal data without assumptions on the presence of critical points. It is shown through different stability estimates that the reconstruction of the coefficient is accurate in areas far from the critical points and deteriorates near these points. The optimality of the stability estimates will be investigated in future works.

8. ACKNOWLEDGEMENTS

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APPENDIX A. THE THREE SPHERES INEQUALITY

As we said before, the proof of the three spheres inequality for the H^1 norm we present here is based on a Carleman inequality. Since the proof for L_q is not so different from the one of a general second order operator in divergence form, we give the proof for this later.

We first consider the following second order operator in divergence form:

$$L = \operatorname{div}(A \nabla \cdot),$$

where $A = (a^{ij})$ is a matrix with coefficients in $W^{1,\infty}(\Omega)$ and there exists $\kappa > 0$ such that

$$(A.1) \quad A(x) \xi \cdot \xi \geq \kappa |\xi|^2 \text{ for any } x \in \Omega \text{ and } \xi \in \mathbb{R}^n.$$

Let $\psi \in C^2(\overline{\Omega})$ having no critical point in $\overline{\Omega}$, we set $\varphi = e^{\lambda \psi}$.

Theorem A.1. (*Carleman inequality*) *There exist three positive constants C , λ_0 and τ_0 , that can depend only on ψ , Ω and a bound of $W^{1,\infty}$ norms of a^{ij} , $1 \leq i, j \leq n$, such that the following inequality holds true:*

$$(A.2) \quad C \int_{\Omega} (\lambda^4 \tau^3 \varphi^3 v^2 + \lambda^2 \tau \varphi |\nabla v|^2) e^{2\tau \varphi} dx \leq \int_{\Omega} (Lw)^2 e^{2\tau \varphi} dx + \int_{\Gamma} (\lambda^3 \tau^3 \varphi^3 v^2 + \lambda \tau \varphi |\nabla v|^2) e^{2\tau \varphi} d\sigma,$$

for all $v \in H^2(\Omega)$, $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$.

Proof. We let $\Phi = e^{-\tau \varphi}$ and $w \in H^2(\Omega)$. Then straightforward computations give

$$Pw = [\Phi^{-1} L \Phi] w = P_1 w + P_2 w + cw.$$

where,

$$P_1 w = aw + \operatorname{div}(A \nabla w),$$

$$P_2 w = B \cdot \nabla w + bw,$$

with

$$a = a(x, \lambda, \tau) = \lambda^2 \tau^2 \varphi^2 |\nabla \psi|_A^2,$$

$$B = B(x, \lambda, \tau) = -2\lambda \tau \varphi A \nabla \psi,$$

$$b = b(x, \lambda, \tau) = -2\lambda^2 \tau \varphi |\nabla \psi|_A^2,$$

$$c = c(x, \lambda, \tau) = -\lambda \tau \varphi \operatorname{div}(A \nabla \psi) + \lambda^2 \tau \varphi |\nabla \psi|_A^2.$$

Here

$$|\nabla\psi|_A = \sqrt{A\nabla\psi \cdot \nabla\psi}.$$

We have

$$(A.3) \quad \int_{\Omega} awB \cdot \nabla w dx = \frac{1}{2} \int_{\Omega} aB \cdot \nabla w^2 dx = -\frac{1}{2} \int_{\Omega} \operatorname{div}(aB)w^2 dx + \frac{1}{2} \int_{\Gamma} aB \cdot \nu w^2 d\sigma$$

and

$$(A.4) \quad \begin{aligned} \int_{\Omega} \operatorname{div}(A\nabla w)B \cdot \nabla w dx &= - \int_{\Omega} A\nabla w \cdot \nabla(B \cdot \nabla w) dx + \int_{\Gamma} B \cdot \nabla w A\nabla w \cdot \nu d\sigma \\ &= - \int_{\Omega} B' \nabla w \cdot A\nabla w dx - \int_{\Omega} \nabla^2 w B \cdot A\nabla w dx + \int_{\Gamma} B \cdot \nabla w \nabla w \cdot \nu d\sigma. \end{aligned}$$

Here, $B' = (\partial_i B_j)$ is the jacobian matrix of B and $\nabla^2 w = (\partial_{ij}^2 w)$ is the hessian matrix of w .

But,

$$\int_{\Omega} B_i \partial_{ij}^2 w a^{jk} \partial_k w dx = - \int_{\Omega} B_i a^{jk} \partial_{ik}^2 w \partial_{ij}^2 w \partial_j w dx - \int_{\Omega} \partial_i B_i a^{jk} \partial_k w \partial_j w dx + \int_{\Gamma} B_i \nu_i a^{jk} \partial_k w \partial_j w d\sigma.$$

Therefore,

$$(A.5) \quad \int_{\Omega} \nabla^2 w B \cdot A\nabla w dx = -\frac{1}{2} \int_{\Omega} [\operatorname{div}(B)A + \tilde{A}\nabla w] \cdot \nabla w dx + \frac{1}{2} \int_{\Gamma} |\nabla w|_A^2 B \cdot \nu d\sigma,$$

with $\tilde{A} = (\tilde{a}^{ij})$,

$$\tilde{a}^{ij} = B \cdot \nabla a^{ij}.$$

It follows from (A.4) and (A.5),

$$(A.6) \quad \begin{aligned} \int_{\Omega} \operatorname{div}(A\nabla w)B \cdot \nabla w dx &= \frac{1}{2} \int_{\Omega} \left(-2B' + \operatorname{div}(B)A + \tilde{A} \right) \nabla w \cdot \nabla w dx \\ &\quad + \int_{\Gamma} B \cdot \nabla w \nabla w \cdot \nu d\sigma - \frac{1}{2} \int_{\Gamma} |\nabla w|_A^2 B \cdot \nu d\sigma. \end{aligned}$$

A new integration by parts yields

$$\int_{\Omega} \operatorname{div}(A\nabla w)bw dx = - \int_{\Omega} b|\nabla w|_A^2 dx - \int_{\Omega} w \nabla b \cdot A\nabla w dx + \int_{\Gamma} bw A\nabla w \cdot \nu d\sigma.$$

This and the following inequality

$$- \int_{\Omega} w \nabla b \cdot A\nabla w dx \geq - \int_{\Omega} (\lambda^2 \varphi)^{-1} |\nabla b|_A^2 w^2 dx - \int_{\Omega} \lambda^2 \varphi |\nabla w|_A^2 dx,$$

imply

$$(A.7) \quad \int_{\Omega} \operatorname{div}(A\nabla w)bw dx \geq - \int_{\Omega} (b + \lambda^2 \varphi) |\nabla w|_A^2 dx - \int_{\Omega} (\lambda^2 \varphi)^{-1} |\nabla b|_A^2 w^2 dx + \int_{\Gamma} bw A\nabla w \cdot \nu d\sigma.$$

Now a combination of (A.3), (A.6) and (A.7) leads

$$(A.8) \quad \int_{\Omega} P_1 w P_2 w dx - \int_{\Omega} c^2 w^2 dx \geq \int_{\Omega} f w^2 dx + \int_{\Omega} F \nabla w \cdot \nabla w dx + \int_{\Gamma} g(w) d\sigma,$$

where,

$$\begin{aligned} f &= -\frac{1}{2} \operatorname{div}(aB) + ab - (\lambda^2 \varphi)^{-1} |\nabla b|_A^2 - c^2, \\ F &= -B' + \frac{1}{2} \left(\operatorname{div}(B)A + \tilde{A} \right) - (b + \lambda^2 \varphi)A, \\ g(w) &= \frac{1}{2} aw^2 B \cdot \nu - \frac{1}{2} |\nabla w|_A^2 B \cdot \nu + B \cdot \nabla w A\nabla w \cdot \nu + bw A\nabla w \cdot \nu. \end{aligned}$$

Using the elementary inequality $(s - t)^2 \geq s^2/2 - t^2$, $s, t > 0$, we obtain

$$\|Pw\|_2^2 \geq (\|P_1w + P_2w\|_2 - \|cw\|_2)^2 \geq \frac{1}{2}\|P_1w + P_2w\|_2^2 - \|cw\|_2^2 \geq \int_{\Omega} P_1wP_2w dx - \int_{\Omega} c^2w^2 dx.$$

In light of (A.8), we get

$$(A.9) \quad \|Lw\|_2^2 \geq \int_{\Omega} fw^2 dx + \int_{\Omega} F\nabla w \cdot \nabla w dx + \int_{\Gamma} g(w) d\sigma.$$

After some straightforward computations, we find that there exist three positive constants C_0, C_1, λ_0 and τ_0 , that can depend only on ψ, Ω and a bound of $W^{1,\infty}$ norms of a^{ij} , $1 \leq i, j \leq n$, such that for all $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$\begin{aligned} f &\geq C_0\lambda^4\tau^3\varphi^3, \\ F\xi \cdot \xi &\geq C_0\lambda^2\tau\varphi|\xi|^2, \text{ for any } \xi \in \mathbb{R}^n, \\ |g(w)| &\leq C_1(\lambda^3\tau^3\varphi^3w^2 + \lambda\tau\varphi|\nabla w|^2). \end{aligned}$$

Hence,

$$C \int_{\Omega} (\lambda^4\tau^3\varphi^3w^2 + \lambda^2\tau\varphi|\nabla w|^2) dx \leq \int_{\Omega} (Pw)^2 dx + \int_{\Gamma} (\lambda^3\tau^3\varphi^3w^2 + \lambda\tau\varphi|\nabla w|^2) d\sigma.$$

As usual, we take $w = \Phi^{-1}v$, $v \in H^1(\Omega)$, in the previous inequality to derive the Carleman estimate (A.2). \square

Remark A.1. *i) We notice that (A.2) is still valid when $v \in H^1(\Omega)$ with $Lv \in L^2(\Omega)$. Also, the inequality (A.2) can be extended to an operator \tilde{L} of the form*

$$\tilde{L} = L + L',$$

where L' is a first order operator with bounded coefficients. For the present case, the constants C, λ_0 and τ_0 in the statement of Theorem A.1 may depend also on a bound of L^∞ norms of the coefficients of L' .

ii) We can substitute L by an operator L_r of the same form, whose coefficients a^{ij} depend on the parameter r , r belonging to some set I . Let L'_r be a first order operator with coefficients depending also on the parameter r . Under the assumption that the coefficients of L_r are uniformly bounded in $W^{1,\infty}(\Omega)$ with respect to r , the coefficients of L'_r are uniformly bounded in $L^\infty(\Omega)$ with respect to the parameter r and the ellipticity condition (A.1) holds with κ independent on r , we can paraphrase the proof of Theorem A.1 and i). We find that (A.2) is true when \tilde{L} is substituted by $\tilde{L} = L_r + L'_r$, with constants C, λ_0 and τ_0 , independent on r . That is we have the following result: there exist three constants C, λ_0 and τ_0 , that can depend on the uniform bound of the coefficients of L_r in $W^{1,\infty}(\Omega)$ and the uniform bound of the coefficients of L'_r in $L^\infty(\Omega)$, such that for any $v \in H^1(\Omega)$ satisfying $\tilde{L}_rv \in L^2(\Omega)$, $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$(A.10) \quad C \int_{\Omega} (\lambda^4\tau^3\varphi^3v^2 + \lambda^2\tau\varphi|\nabla v|^2) e^{2\tau\varphi} dx \leq \int_{\Omega} (\tilde{L}_rv)^2 e^{2\tau\varphi} dx + \int_{\Gamma} (\lambda^3\tau^3\varphi^3v^2 + \lambda\tau\varphi|\nabla v|^2) e^{2\tau\varphi} d\sigma.$$

Lemma A.1. *(Three spheres inequality) There exist $C > 0$ and $0 < s < 1$, that only depend on a bound of $W^{1,\infty}$ norms of the coefficients of L and a bound of L^∞ norms of the coefficients of L' , such that, for all $v \in H^1(\Omega)$ satisfying $\tilde{L}v = 0$ in Ω , $y \in \Omega$ and $0 < r < \frac{1}{3}\text{dist}(y, \Gamma)$,*

$$r\|v\|_{H^1(B(y,2r))} \leq C\|v\|_{H^1(B(y,r))}^s \|v\|_{H^1(B(y,3r))}^{1-s}.$$

Proof. Let $v \in H^1(\Omega)$ satisfying $\tilde{L}v = 0$, set $B(i) = B(0, i)$, $i = 1, 2, 3$ and $r_0 = \frac{1}{3}\text{diam}(\Omega)$. Fix $y \in \Omega$ and $0 < r < \frac{1}{3}\text{dist}(y, \Gamma) (\leq r_0)$. Let

$$w(x) = v(rx + y), \quad x \in B(3),$$

Clearly, $\tilde{L}_rw = 0$, with an operator \tilde{L}_r as in ii) of Remark A.1.

Let $\chi \in C_c^\infty(U)$ satisfying $0 \leq \chi \leq 1$ and $\chi = 1$ in K , with

$$U = \{x \in \mathbb{R}^n; 1/2 < |x| < 3\}, \quad K = \{x \in \mathbb{R}^n; 1 \leq |x| \leq 5/2\}.$$

We get by applying (A.10) to χw , with U in place of Ω , $\lambda \geq \lambda_0$ and $\tau \geq \tau_0$,

$$(A.11) \quad C \int_{B(2) \setminus B(1)} (\lambda^4 \tau^3 \varphi^3 w^2 + \lambda^2 \tau \varphi |\nabla w|^2) e^{2\tau\varphi} dx \leq \int_{B(3)} (\tilde{L}_r(\chi w))^2 e^{2\tau\varphi} dx$$

From $\tilde{L}_r w = 0$ and the properties of χ , we obtain in a straightforward manner that

$$\text{supp}(\tilde{L}_r(\chi w)) \subset \{1/2 \leq |x| \leq 1\} \cup \{5/2 \leq |x| \leq 3\}$$

and

$$(\tilde{L}_r(\chi w))^2 \leq \Lambda(w^2 + |\nabla w|^2),$$

where $\Lambda = \Lambda(r_0)$ is independent on r . Therefore, fixing λ and shortening τ_0 if necessary, (A.11) implies, for $\tau \geq \tau_0$,

$$(A.12) \quad C \int_{B(2)} (w^2 + |\nabla w|^2) e^{2\tau\varphi} dx \leq \int_{B(1)} (w^2 + |\nabla w|^2) e^{2\tau\varphi} dx + \int_{\{5/2 \leq |x| \leq 3\}} (w^2 + |\nabla w|^2) e^{2\tau\varphi} dx.$$

Let us now specify φ . The choice of $\varphi(x) = -|x|^2$ (which is with no critical point in U) in (A.12) gives, for $\tau \geq \tau_0$,

$$(A.13) \quad C \int_{B(2)} (w^2 + |\nabla w|^2) dx \leq e^{\alpha\tau} \int_{B(1)} (w^2 + |\nabla w|^2) dx + e^{-\beta\tau} \int_{B(3)} (w^2 + |\nabla w|^2) dx,$$

where

$$\begin{aligned} \alpha &= (1 - e^{-2\lambda}) \\ \beta &= 2 \left(e^{-2\lambda} - e^{-\frac{5}{2}\lambda} \right). \end{aligned}$$

We introduce the following temporary notations

$$\begin{aligned} P &= \int_{B(1)} (w^2 + |\nabla w|^2) dx, \\ Q &= C \int_{B(2)} (w^2 + |\nabla w|^2) dx, \\ R &= \int_{B(3)} (w^2 + |\nabla w|^2) dx. \end{aligned}$$

Then (A.13) reads

$$(A.14) \quad Q \leq e^{\alpha\tau} P + e^{-\beta\tau} R, \quad \tau \geq \tau_0.$$

Let

$$\tau_1 = \frac{\ln(R/P)}{\alpha + \beta}.$$

If $\tau_1 \geq \tau_0$, then $\tau = \tau_1$ in (A.14) yields

$$(A.15) \quad Q \leq P^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}.$$

If $\tau_1 < \tau_0$, $R < e^{(\alpha+\beta)\tau_0} P$ and then

$$(A.16) \quad Q \leq R = R^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}} \leq e^{\alpha\tau_0} P^{\frac{\alpha}{\alpha+\beta}} R^{\frac{\beta}{\alpha+\beta}}.$$

Summing up, we get that one inequalities (A.15) and (A.16) holds. That is, we have in terms of our original notations

$$\|w\|_{H^1(B(2))} \leq C \|w\|_{H^1(B(1))}^s \|w\|_{H^1(B(3))}^{1-s}.$$

We complete the proof by noting that

$$(A.17) \quad c_0 r^{1-n/2} \|v\|_{H^1(B(y,ir))} \leq \|v\|_{H^1(B(y,ir))} \leq c_1 r^{-n/2} \|v\|_{H^1(B(y,ir))}, \quad i = 1, 2, 3,$$

where

$$c_0 = \min(1, r_0), \quad c_1 = \max(1, r_0).$$

Finally, we observe that estimate (A.17) can be obtained in a simple way after making a change of variable. \square

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